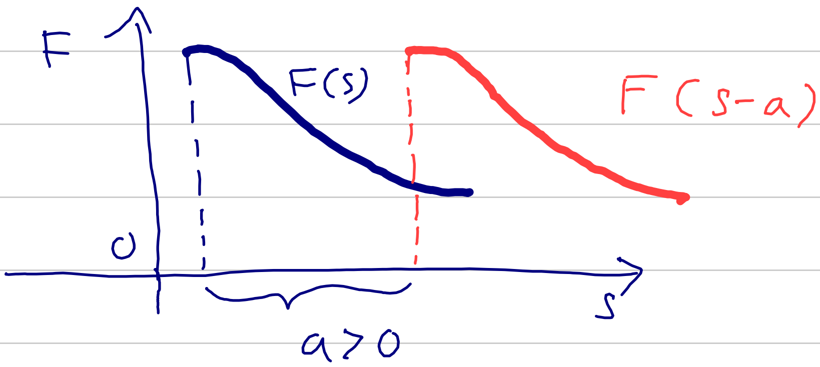
7.3. OPERATIONAL PROPERTIES I.

1. TRANSLATION ON THE s-AXIS



Theorem (First Translation Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Proof

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a). \quad \square$$

If we consider s a real variable, then the graph of $F(s-a)$ is the graph of $F(s)$ shifted on s-axis by the amount $|a|$. If $a>0$, we shift to the right; if $a<0$, we shift to the left.

We sometimes write

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\big|_{s\to s-a}.$$

Example: Evaluate (a) $\mathcal{L}\{e^{6t}t^4\}$

(b) $\mathcal{L}\{e^{-4t}\cos 3t\}$.

Solution: (a) $\mathcal{L}\{e^{6t}t^4\} = \mathcal{L}\{t^4\}\Big|_{s\to s-6}$

$$= \frac{4!}{s^5}\Big|_{s\to s-6} = \frac{24}{(s-6)^5}.$$

(b) $\mathcal{L}\{e^{-4t}\cos 3t\} = \mathcal{L}\{\cos 3t\}\Big|_{s\to s+4}$

$$= \frac{s}{s^2+9}\Big|_{s\to s+4} = \frac{s+4}{(s+4)^2+9}. \quad \square$$

Corollary: $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Example: Evaluate a) $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$

b) $\mathcal{L}^{-1}\left\{\frac{s/2+5/2}{s^2+4s+6}\right\}$.

Solution

(a) $$\frac{2s+5}{(s-3)^2} = \frac{2s-6}{(s-3)^2} + \frac{11}{(s-3)^2}$$

$$= \frac{2}{(s-3)} + \frac{11}{(s-3)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\Big|_{s\to s-3}\right\}$$

$$+ 11\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s-3}\right\}.$$

We have $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2e^{3t} + 11e^{3t}t.$$

(b) $$\frac{s/2 + 5/2}{s^2+4s+6} = \frac{s/2 + 5/2}{(s+2)^2+2}$$

$$= \frac{(s+2)/2 - 1 + 5/2}{(s+2)^2+2}$$

$$= \frac{1}{2}\frac{(s+2)}{(s+2)^2+2} + \frac{3}{2}\frac{1}{(s+2)^2+2}$$

$$= \frac{1}{2}\frac{(s+2)}{(s+2)^2+2} + \frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{(s+2)^2+2}.$$

$$\mathcal{L}^{-1}\left\{\frac{s/2+5/2}{s^2+4s+6}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+2}\right\} + \frac{3}{2\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\Big|_{s\to s+2}\right\} + \frac{3}{2\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\Big|_{s\to s+2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{3}{2\sqrt{2}}e^{-2t}\sin\sqrt{2}t.$$

Example: Solve $y'' - 6y' + 9y = t^2e^{3t}$

$y(0) = 2, \quad y'(0) = 17.$

Solution:

Denote $Y(s) = \mathcal{L}\{y(t)\}$.

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - 6[sY(s) - y(0)] + 9\,Y(s) = \frac{2}{(s-3)^3}$$

$$\Leftrightarrow \quad (s^2 - 6s + 9)\,Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$\Leftrightarrow \quad Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$\Leftrightarrow \quad Y(s) = \frac{2}{(s-3)} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$\Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\}$$

$$= 2.e^{3t} + 11\,e^{3t}t + \frac{2}{4!}\,e^{3t}t^4. \quad \square$$

<u>Example</u>: Solve $y'' + 4y' + 6y = 1 + e^{-t}$

$y(0) = 0, \; y'(0) = 0$

<u>Solution</u>:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{1 + e^{-t}\}$$

$$\Leftrightarrow$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4[sY(s) - y(0)] + 6\,Y(s) = \frac{1}{s} + \frac{1}{s+1}.$$

$$\Leftrightarrow \quad (s^2 + 4s + 6)\,Y(s) = \frac{2s+1}{s(s+1)}$$

$$\Rightarrow Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)}$$

$$= \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2+4s+6}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\}$$

$$- \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{2}{3\sqrt{2}}e^{-2t}\sin\sqrt{2}t.$$

$\square$

2. TRANSLATION ON THE t-AXIS:

Defn. The **unit-step function** $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a. \end{cases}$$

Let $f(t)$ be a piecewise-define function

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$$

Then

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

Similar, a function of the type

$$f(t) = \begin{cases} 0, & 0 \leq t < a \text{ or } t \geq b \\ g(t), & a \leq t \leq b \end{cases}$$

can be written
$$f(t) = g(t)\left[\mathcal{U}(t-a) - \mathcal{U}(t-b)\right].$$

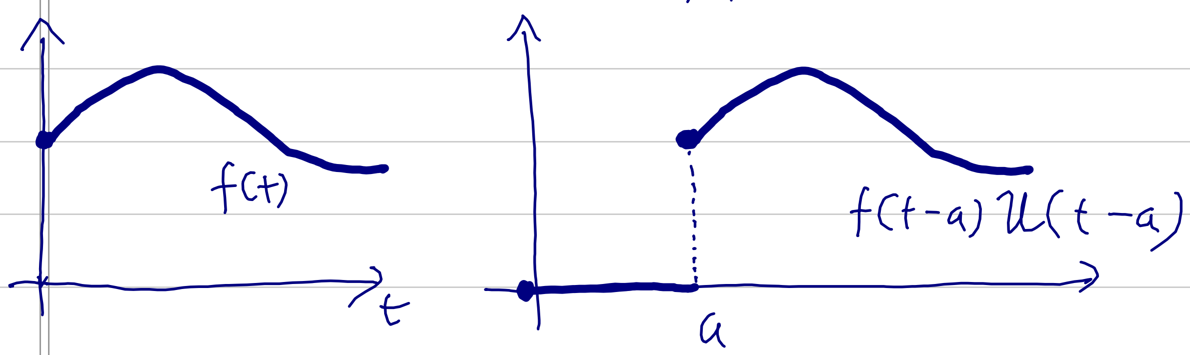
Example: Express $f(t) = \begin{cases} 20t & , 0 \le t < 5 \\ 0 & , 5 \le t \end{cases}$

in terms of unit step functions.

Solution: $f(t) = 20t - 20t\,\mathcal{U}(t-5).$

Theorem (Second Translation Theorem)

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then
$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s).$$



Proof:

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{=0 \text{ when } t<a}\,dt$$

$$+ \int_a^\infty e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{=1 \text{ when } t\ge a}\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_0^\infty e^{-s(v+a)} f(v)\, dv$$

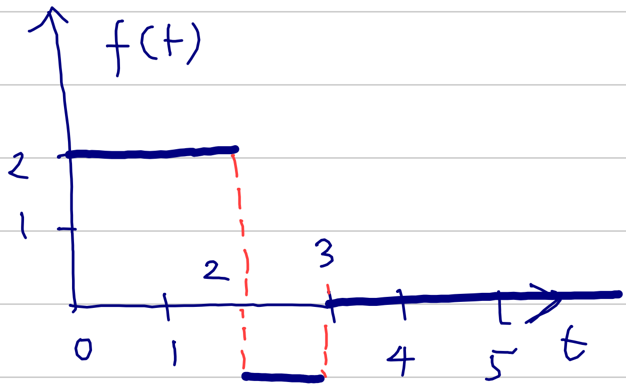
$(v = t - a;\ dv = dt)$

$$= e^{-as} \int_0^\infty e^{-sv} f(v)\, dv = e^{-as} F(s). \quad \square$$

We have:

$$\mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{1\} = \frac{e^{-as}}{s}.$$

Example:



Find $\mathcal{L}\{f(t)\}$.

Solution:

$$f(t) = 2 - 2\mathcal{U}(t-2) + g(t)$$

$$g(t) = \begin{cases} -1, & 2 \leq t < 3 \\ 0, & \text{otherwise.} \end{cases}$$

$$g(t) = (-1)\left[\mathcal{U}(t-2) - \mathcal{U}(t-3)\right]$$

$$\Rightarrow \quad f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}. \quad \square$$

Corollary: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a).$

Example Evaluate

(a) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-4}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{s \cdot e^{-\pi s/2}}{s^2+9}\right\}$

Solution: (a) $a = 2$, $F(s) = \frac{1}{s-4}$, $\mathcal{L}^{-1}\{F(s)\} = e^{4t}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-4}\right\} = e^{4(t-2)}\,\mathcal{U}(t-2)$

(b) $a = \frac{\pi}{2}$, $F(s) = \frac{s}{s^2+9}$, $\mathcal{L}^{-1}\{F(s)\} = \cos 3t.$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\,e^{-\pi s/2}\right\} = \cos 3\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right).$

$\square$

ALTERNATIVE FORM OF THE SECOND TRANSLATION THEOREM:

$$\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}$$

Example: Evaluate $\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\}.$

Solution:

$g(t) = \cos t$ and $a = \pi$, $g(t+\pi) = \cos(t+\pi)$
$= -\cos t.$

$\Rightarrow \mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\} = -e^{-\pi s}\mathcal{L}\{\cos t\}$

$= -\frac{s}{s^2+1}\,e^{-\pi s}.$ $\square$

Example: Solve $y' + y = f(t)$, $y(0) = 5$, where

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 3\cos t, & t \geq \pi. \end{cases}$$

Solution: $f(t) = 3\cos t \, \mathcal{U}(t-\pi)$

$$\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t \, \mathcal{U}(t-\pi)\}$$

$$\Leftrightarrow sY(s) - y(0) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s} \quad (\text{previous Example})$$

$$\Leftrightarrow Y(s) = \frac{5}{s+1} - \frac{3s}{(s+1)(s^2+1)}e^{-\pi s}$$

$$\Leftrightarrow Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]$$

We have

$$\mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} = 5e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s+1}\right\} = e^{(t-\pi)}\mathcal{U}(t-\pi)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\,\mathcal{U}(t-\pi)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}\cdot s}{s^2+1}\right\} = \cos(t-\pi)\,\mathcal{U}(t-\pi).$$

$$\Rightarrow y(t) = 5e^{-t} + \frac{3}{2}\left(e^{-(t-\pi)} + \sin t + \cos t\right)\mathcal{U}(t-\pi)$$

$$= \begin{cases} 5e^{-t}, & 0 \leq t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-t+\pi} + \frac{3}{2}\sin t + \frac{3}{2}\cos t, & t \geq \pi. \end{cases}$$